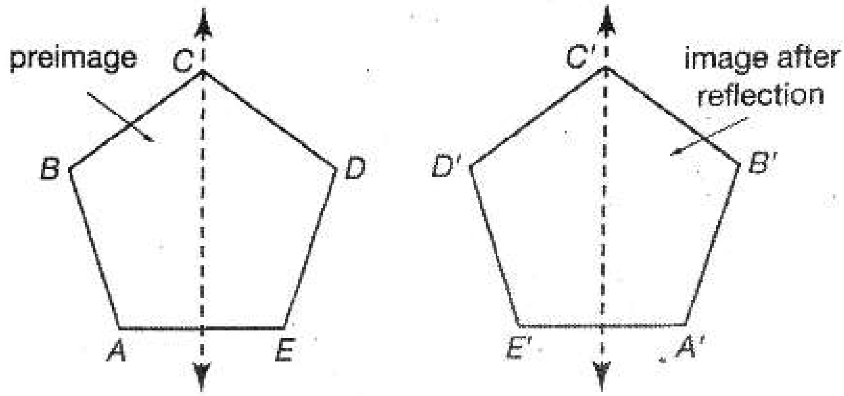
LESSON 5

Symmetry and Sequences of Transformations

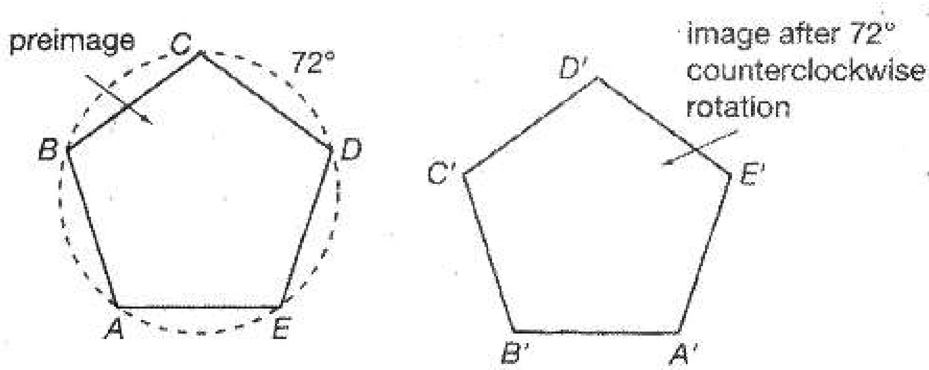
Types of Symmetry

UNDERSTAND A **regular polygon** is a polygon with all sides equal in length and all angles equal in measure. If a regular polygon has n sides, then it also has n **lines of symmetry.** When you reflect a figure over a line of symmetry, the image is identical to and in the same location as the original preimage. When this happens, we say that the reflection maps the figure onto itself. This type of symmetry is called **line symmetry** or **reflectional symmetry.**

The regular pentagon shown below has 5 lines of symmetry. One of them is the perpendicular bisector of $\overline{AE}$. If this pentagon is reflected across the dashed line, point B is carried onto point D and vice versa, point A is carried onto point E and vice versa, and point C maps onto itself because it is on the line of reflection. The image is identical to its preimage.



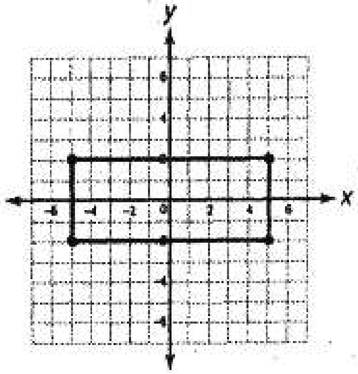
The pentagon also has **rotational symmetry**. A figure that has rotational symmetry will map onto itself more than once during a 360° turn. Notice that if a circle is drawn through all five vertices, the circle is divided into five equal-length arcs. To find the measure of each arc, divide 360° by the number of sides. $360° \div 5 = 72°$. Rotating the pentagon 72 degrees maps it onto itself.



If you were to rotate the pentagon another 72°, which is a 144° rotation from the original preimage, you would produce the same figure again. You can do this 3 more times. In general, a regular polygon with n sides will map onto itself n times during a 360° turn.

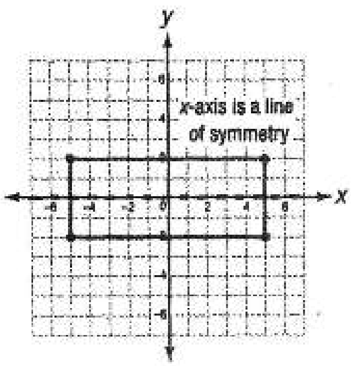
Connect

Describe two ways in which the rectangle graphed below could be mapped back onto itself.



1 Look for line symmetry.

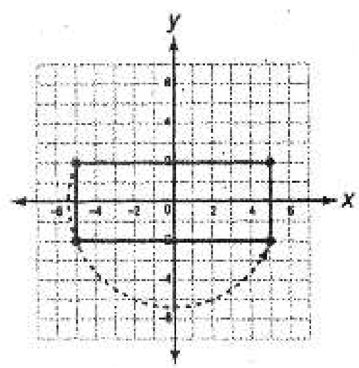
If you reflect the rectangle over the *x*-axis, then the top half maps onto the bottom half and vice versa.



▶ A reflection across the *x*-axis maps the rectangle back onto itself.

2 Look for rotational symmetry.

If you rotate the rectangle 180°, the upper left corner maps onto the lower right corner and vice versa. The resulting figure is identical to the original rectangle.



▶ A rotation of 180° around the origin will map the rectangle back onto itself.

DISCUSS

Is there a third way to map the rectangle onto itself in one step? If so, describe it. If not, explain why not.

Transformations in Sequence

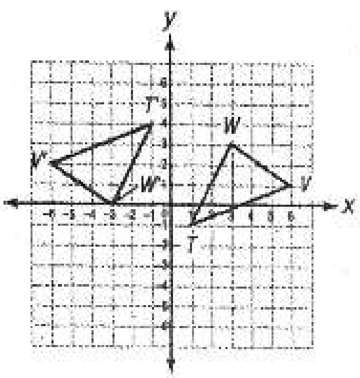
UNDERSTAND Sometimes, more than one transformation is needed to produce a particular image from a given preimage. There are often multiple sequences that can produce a given image. Whether a single rigid motion or a sequence of rigid motions is used, the final image is always congruent to the original preimage.

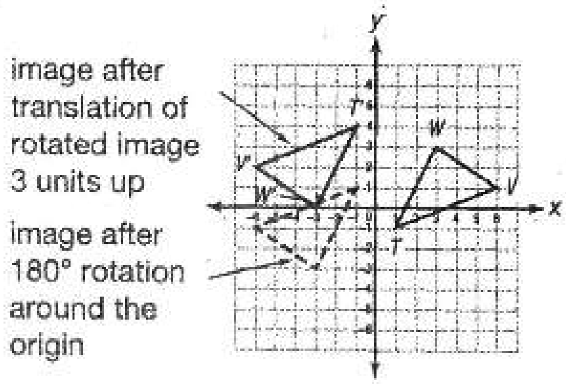
To determine the necessary sequence of transformations, compare the preimage to the image much as you have done for individual transformations. If the orientation of the figure has changed, then a reflection or rotation has probably taken place. Once the figures have the same orientation, apply translations in the sequence to align the figures.

Consider $\triangle TVW$ and $\triangle T'V'W'$ below. The image has a different orientation than the preimage. Examining the shape and the placement of the vertices shows that the image appears to be a 180° rotation of the preimage. Try rotating $\triangle TVW$ by 180° around the origin and comparing the image to $\triangle T'V'W'$.

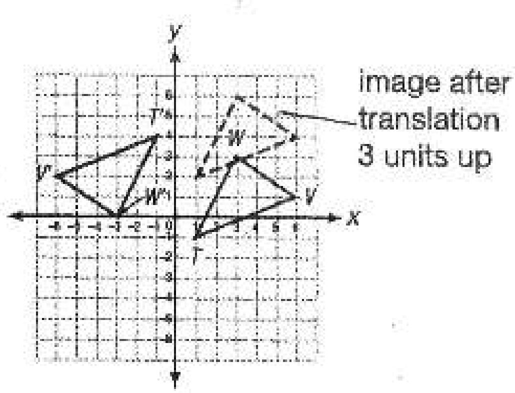
After the rotation, the orientation of the image matches the orientation of $\triangle T'V'W'$, but each point on the image is 3 units below $\triangle T'V'W'$. Apply a translation to align the figures.

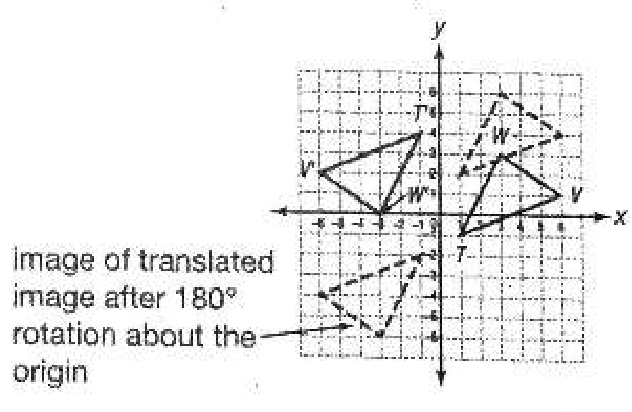
So, a 180° rotation of $\triangle TVW$ about the origin followed by a translation of 3 units up will produce $\triangle T'V'W'$.





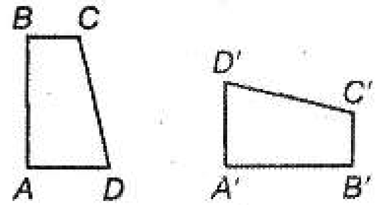
The order in which you apply transformations does not always matter, but in this case, it does. If you translate $\triangle TVW$ up 3 units and *then* rotate, you end up with a congruent figure with the correct orientation but in the wrong place. You would still need to perform another translation (6 units up) to map your image onto $\triangle T'V'W'$.





Connect

Describe a sequence of transformations that could be used to map trapezoid *ABCD* onto trapezoid *A'B'C'D'*.



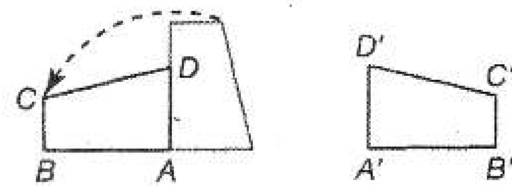
1 Compare the size, shape, and orientation of the image and the preimage.

The trapezoids appear to have the same size and shape, so a series of rigid motions should produce one from the other.

ABCD is taller than it is wide, and *A'B'C'D'* is wider than it is tall, so the two figures have different orientations. The sequence may include a rotation.

2 Rotate the preimage and compare to *A'B'C'D'*.

In the image, side *A'B'* is on the bottom and *C'D'* is on top, so rotate *ABCD* 90° counterclockwise around point *A*.



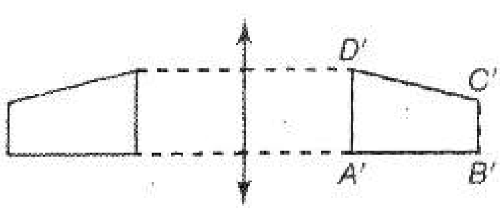
On the blue image, begin at vertex *A* and read the vertices clockwise around the figure: *A* - *B* - *C* - *D*.

Do the same with the green image, beginning at vertex *A'*: *A'* - *D'* - *C'* - *B'*.

The corresponding vertices are not in the same order. This means that a reflection must be part of the sequence.

3 Reflect the rotated image and compare to *A'B'C'D'*.

Place a vertical line of reflection halfway between points *D* and *D'*. Reflect the rotated image over this line.



▶ A rotation of 90° counterclockwise followed by a horizontal reflection produces *A'B'C'D'* from *ABCD*.

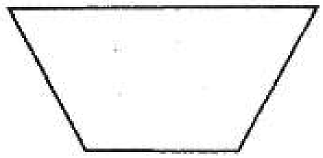
TRY

Find other sequences that map trapezoid *ABCD* onto trapezoid *A'B'C'D'*.

Practice

Determine if the given figure has *rotational symmetry, line symmetry, both,* or *neither.*

1. isosceles trapezoid



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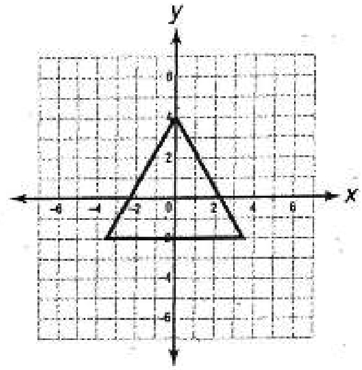
2. equilateral triangle



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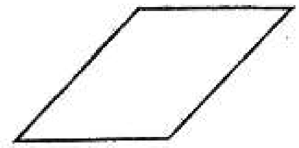
3. parallelogram



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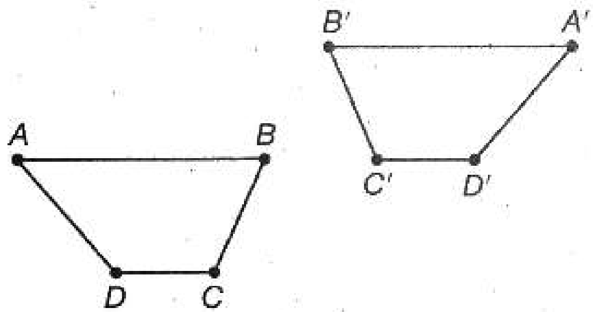
> **HINT** A figure has rotational symmetry if some turn of less than 360° maps it back onto itself.

Choose the best answer.

4. Which sequence of rigid motions could be used to transform trapezoid *ABCD* to trapezoid *A″B″C″D″*?



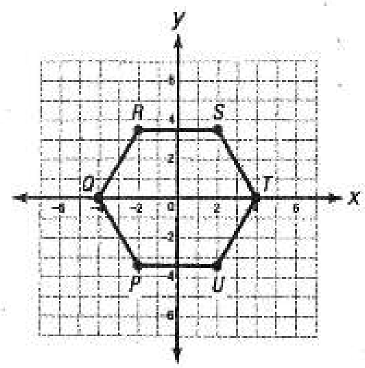
A. translation of trapezoid *ABCD* up and to the left

B. translation of trapezoid *ABCD* up and to the right

C. a 180° rotation of trapezoid *ABCD* followed by a horizontal reflection

D. a horizontal reflection of trapezoid *ABCD* followed by a translation up

5. Which does **not** describe a way to map regular hexagon *PQRSTU* back onto itself?



A. reflect it across the *x*-axis

B. reflect it across the *y*-axis

C. rotate it 60°

D. rotate it 90°

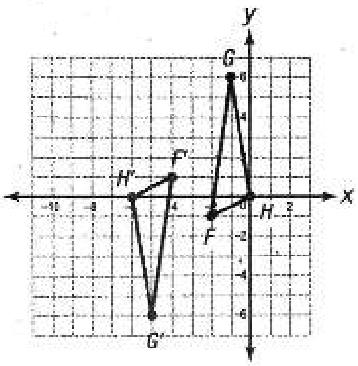
Write *true* or *false* for each statement.

6. Any figure will map back onto itself after a 360° turn about its center. ____________

7. Any figure that has line symmetry must also have rotational symmetry. ____________

Solve.

8. **SHOW** Describe a sequence of two transformations that could map $\triangle FGH$ onto $\triangle F'G'H'$.



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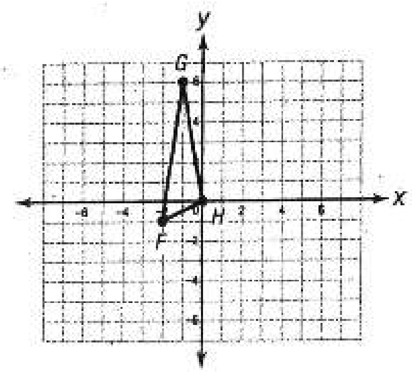
9. **EXPLAIN** Reverse the order of the transformations in your sequence from question 8 and draw the image on the plane below. Does this affect the final image produced? Explain.



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10. **DRAW** A certain quadrilateral can be mapped onto itself after 90°, 180°, and 270° rotations about its center. It can also be mapped onto itself by a horizontal reflection and a vertical reflection. Name the quadrilateral that fits that description and draw it below.

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