3.1 Sequences of Transformations

Essential Question: What happens when you apply more than one transformation to a figure?

Resource Locker

Explore Combining Rotations or Reflections

A transformation is a function that takes points on the plane and maps them to other points on the plane. Transformations can be applied one after the other in a sequence where you use the image of the first transformation as the preimage for the next transformation.

Find the image for each sequence of transformations.

(A) Using geometry software, draw a triangle and label the vertices A, B, and C. Then draw a point outside the triangle and label it P.

Rotate $\triangle ABC$ 30° around point P and label the image as $\triangle A'B'C'$. Then rotate $\triangle A'B'C'$ 45° around point P and label the image as $\triangle A''B''C''$. Sketch your result.

(B) Make a conjecture regarding a single rotation that will map $\triangle ABC$ to $\triangle A''B''C''$. Check your conjecture, and describe what you did.

(C) Using geometry software, draw a triangle and label the vertices D, E, and F. Then draw two intersecting lines and label them j and k.

Reflect $\triangle DEF$ across line j and label the image as $\triangle D'E'F'$. Then reflect $\triangle D'E'F'$ across line k and label the image as $\triangle D''E''F''$. Sketch your result.

(D) Consider the relationship between $\triangle DEF$ and $\triangle D''E''F''$. Describe the single transformation that maps $\triangle DEF$ to $\triangle D''E''F''$. How can you check that you are correct?

Reflect

1. Repeat Step A using other angle measures. Make a conjecture about what single transformation will describe a sequence of two rotations about the same center.

2. Make a conjecture about what single transformation will describe a sequence of three rotations about the same center.

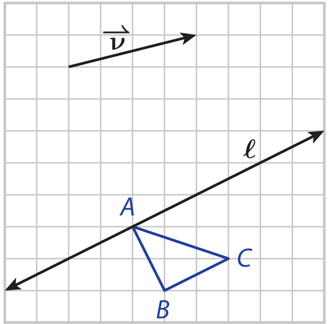
3. **Discussion** Repeat Step C, but make lines j and k parallel instead of intersecting. Make a conjecture about what single transformation will now map $\triangle DEF$ to $\triangle D''E''F''$. Check your conjecture and describe what you did.

Explain 1 Combining Rigid Transformations

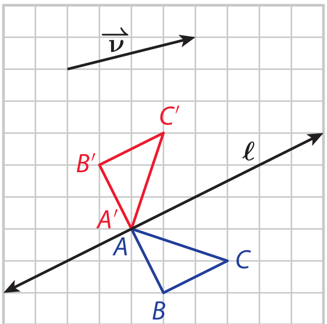
In the Explore, you saw that sometimes you can use a single transformation to describe the result of applying a sequence of two transformations. Now you will apply sequences of rigid transformations that cannot be described by a single transformation.

Example 1 **Draw the image of $\triangle ABC$ after the given combination of transformations.**

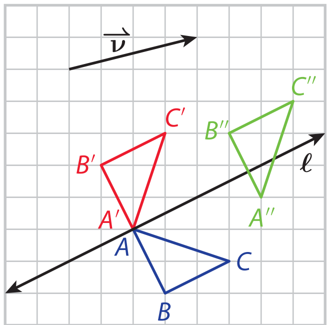
(A) Reflection over line ℓ then translation along $\vec{v}$



Step 1 Draw the image of $\triangle ABC$ after a reflection across line ℓ. Label the image $\triangle A'B'C'$.



Step 2 Translate $\triangle A'B'C'$ along $\vec{v}$. Label this image $\triangle A''B''C''$.

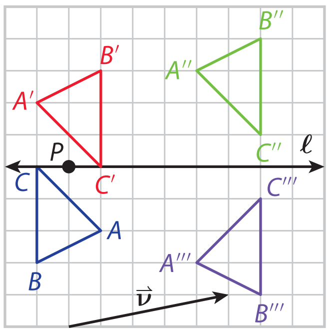


(B) 180° rotation around point P, then translation along $\vec{v}$, then reflection across line ℓ

Apply the rotation. Label the image $\triangle A'B'C'$.

Apply the translation to $\triangle A'B'C'$. Label the image $\triangle A''B''C''$.

Apply the reflection to $\triangle A''B''C''$. Label the image $\triangle A'''B'''C'''$.



Reflect

4. Are the images you drew for each example the same size and shape as the given preimage? In what ways do rigid transformations change the preimage?

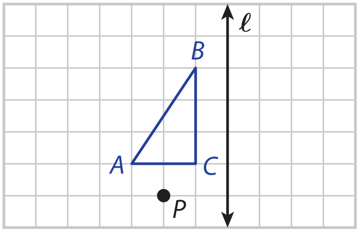
5. Does the order in which you apply the transformations make a difference? Test your conjecture by performing the transformations in Part B in a different order.

6. For Part B, describe a sequence of transformations that will take $\triangle A''B''C''$ back to the preimage.

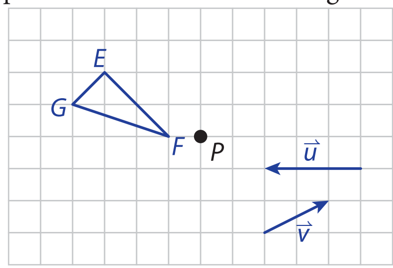
Your Turn

Copy ΔABC on grid paper. Then draw the image of the triangle after the given combination of transformations.

7. Reflection across ℓ then 90° rotation around point P



8. Translation along $\vec{v}$ then 180° rotation around point P then translation along $\vec{u}$

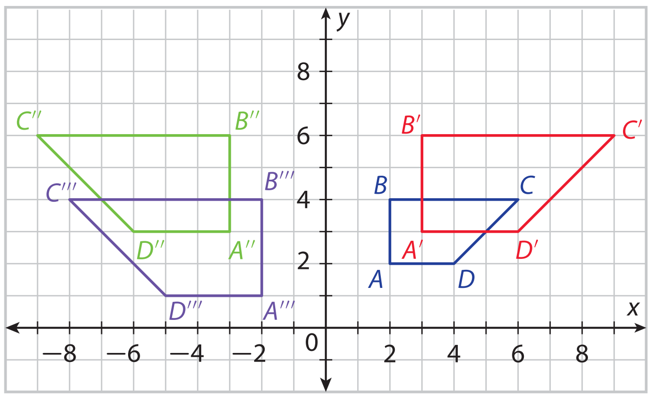


Explain 2 Combining Nonrigid Transformations

Example 2 **Draw the image of the figure in the plane after the given combination of transformations.**

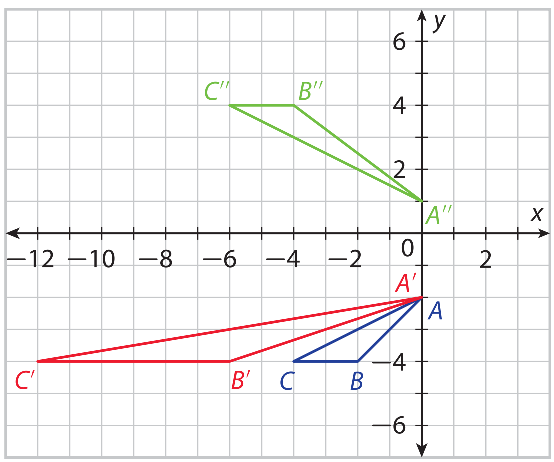
A $(x, y) \rightarrow \left(\frac{3}{2}x, \frac{3}{2}y\right) \rightarrow (-x, y) \rightarrow (x + 1, y - 2)$

1. The first transformation is a dilation by a factor of $\frac{3}{2}$. Apply the dilation. Label the image $A'B'C'D'$.
2. Apply the reflection of $A'B'C'D'$ across the y-axis. Label this image $A''B''C''D''$.
3. Apply the translation of $A''B''C''D''$. Label this image $A'''B'''C'''D'''$.



B $(x, y) \rightarrow (3x, y) \rightarrow \left(\frac{1}{2}x, -\frac{1}{2}y\right)$

1. The first transformation is a horizontal stretch by a factor of 3.
Apply the stretch. Label the image $\triangle A'B'C'$.
2. The second transformation is a dilation by a factor of $\frac{1}{2}$ combined with a reflection.
3. Apply the transformation to $\triangle A'B'C'$. Label the image $\triangle A''B''C''$.



Reflect

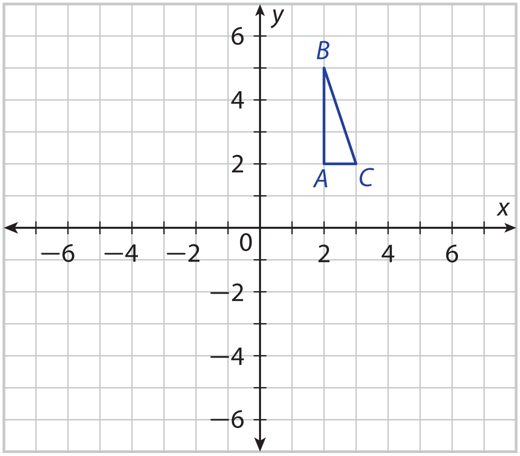
9. If you dilated a figure by a factor of 2, what transformation could you use to return the figure back to its preimage? If you dilated a figure by a factor of 2 and then translated it right 2 units, write a sequence of transformations to return the figure back to its preimage.

10. A student is asked to reflect a figure across the y-axis and then vertically stretch the figure by a factor of 2. Describe the effect on the coordinates. Then write one transformation using coordinate notation that combines these two transformations into one.

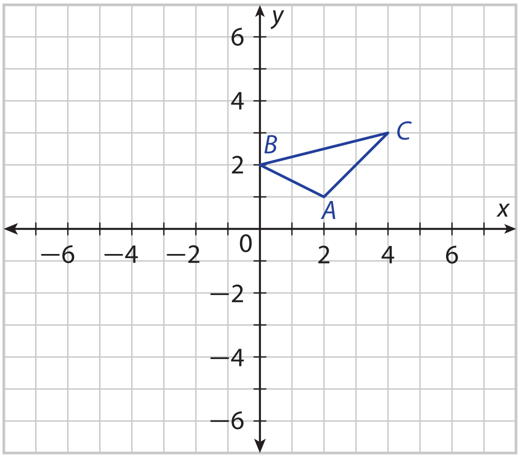
Your Turn

Draw ΔABC on a coordinate grid. Then draw the image of the triangle after the given combination of transformations.

11. $(x, y) \rightarrow (x - 1, y - 1) \rightarrow (3x, y) \rightarrow (-x, -y)$



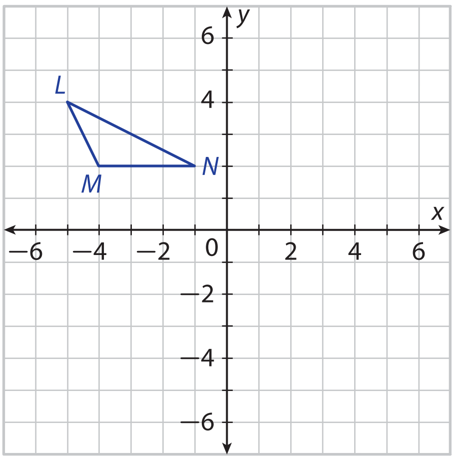
12. $(x, y) \rightarrow \left(\frac{3}{2}x, -2y\right) \rightarrow (x - 5, y + 4)$



Explain 3 Predicting the Effect of Transformations

Example 3 **Predict the result of applying the sequence of transformations to the given figure.**

(A) $\triangle LMN$ is translated along the vector $\langle -2, 3\rangle$, reflected across the y-axis, and then reflected across the x-axis.



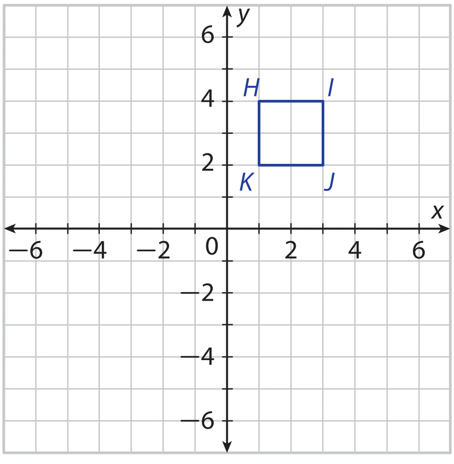
Predict the effect of the first transformation: A translation along the vector $\langle -2, 3\rangle$ will move the figure left 2 units and up 3 units. Since the given triangle is in Quadrant II, the translation will move it further from the x- and y-axes. It will remain in Quadrant II.

Predict the effect of the second transformation: Since the triangle is in Quadrant II, a reflection across the y-axis will change the orientation and move the triangle into Quadrant I.

Predict the effect of the third transformation: A reflection across the x-axis will again change the orientation and move the triangle into Quadrant IV. The two reflections are the equivalent of rotating the figure 180° about the origin.

The final result will be a triangle the same shape and size as $\triangle LMN$ in Quadrant IV. It has been rotated 180° about the origin and is farther from the axes than the preimage.

(B) Square $HIJK$ is rotated 90° clockwise about the origin and then dilated by a factor of 2, which maps $(x, y) \rightarrow (2x, 2y)$.



Predict the effect of the first transformation: A 90° clockwise rotation will map it to Quadrant IV. Due to its symmetry, it will appear to have been translated, but will be closer to the x-axis than it is to the y-axis.

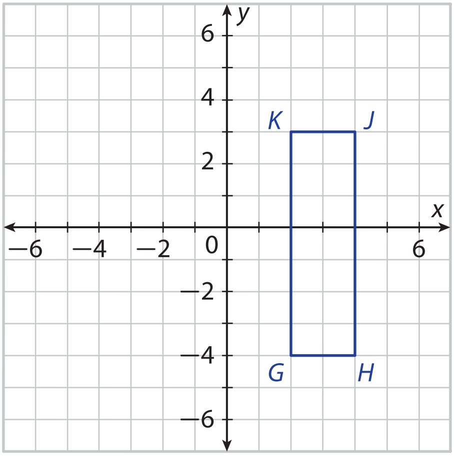
Predict the effect of the second transformation: A dilation by a factor of 2 will double the side lengths of the square. It will also be further from the origin than the preimage.

The final result will be a square in Quadrant 4 with side lengths twice as long as the side lengths of the original. The image is further from the origin than the preimage.

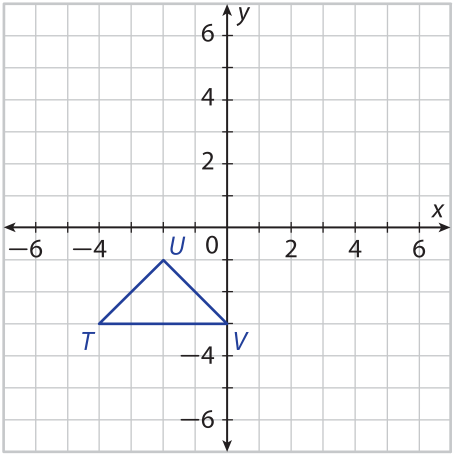
Your Turn

Predict the result of applying the sequence of transformations to the given figure.

13. Rectangle $GHJK$ is reflected across the y-axis and translated along the vector $\langle 5, 4 \rangle$.



14. $\triangle TUV$ is horizontally stretched by a factor of $\frac{3}{2}$, which maps $(x, y) \rightarrow \left(\frac{3}{2}x, y\right)$, and then translated along the vector $\langle 2, 1 \rangle$.



Elaborate

15. Discussion How many different sequences of rigid transformations do you think you can find to take a preimage back onto itself? Explain your reasoning.

16. Is there a sequence of a rotation and a dilation that will result in an image that is the same size and position as the preimage? Explain your reasoning.

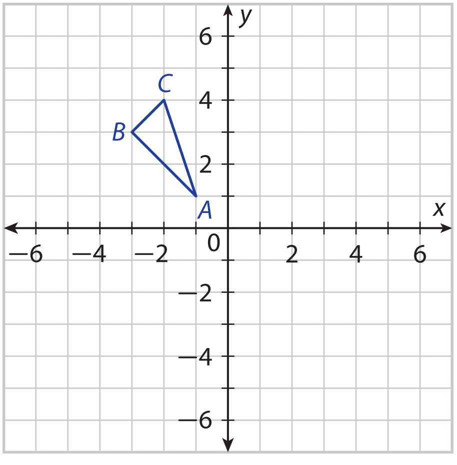
17. Essential Question Check-In In a sequence of transformations, the order of the transformations can affect the final image. Describe a sequence of transformations where the order does not matter. Describe a sequence of transformations where the order does matter.

Evaluate: Homework and Practice

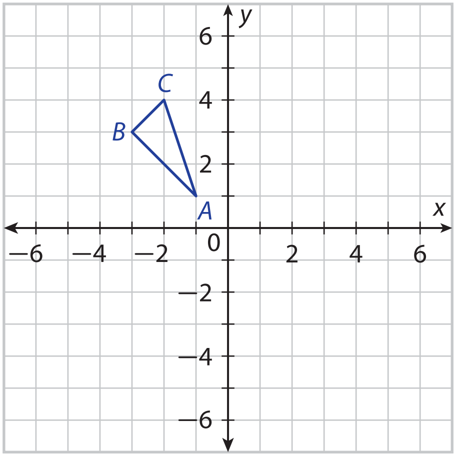
- Online Homework
- Hints and Help
- Extra Practice

Copy ΔABC on a coordinate grid. Then draw and label the final image of $\triangle ABC$ after the given sequence of transformations.

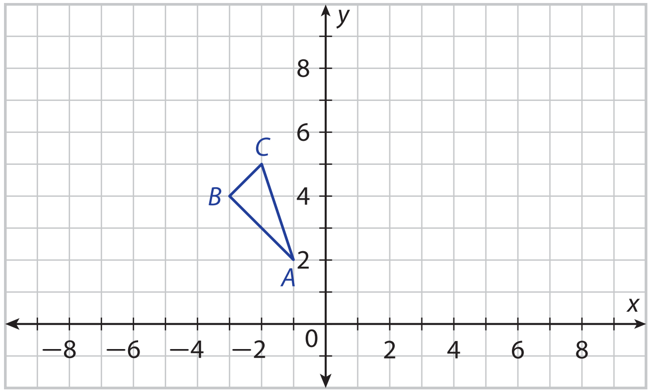
1. Reflect $\triangle ABC$ over the y-axis and then translate by $\langle 2, -3 \rangle$.



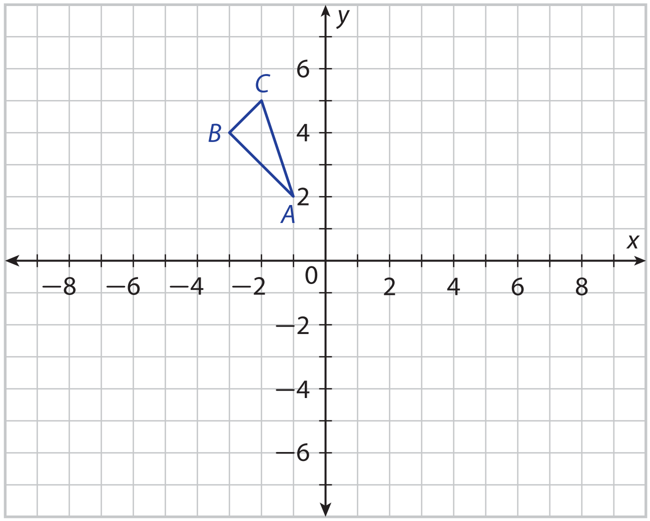
2. Rotate $\triangle ABC$ 90 degrees clockwise about the origin and then reflect over the x-axis.



3. Translate $\triangle ABC$ by $\langle 4, 4 \rangle$, rotate 90 degrees counterclockwise around A, and reflect over the y-axis.

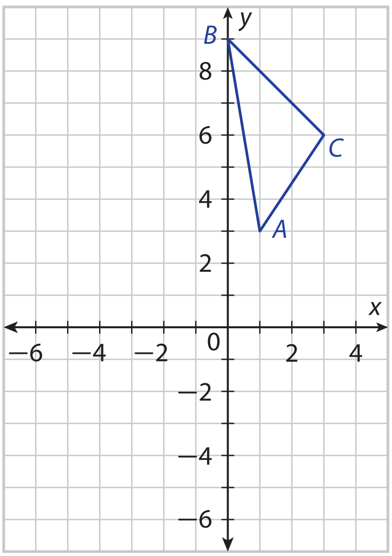


4. Reflect $\triangle ABC$ over the x-axis, translate by $\langle -3, -1 \rangle$, and rotate 180 degrees around the origin.

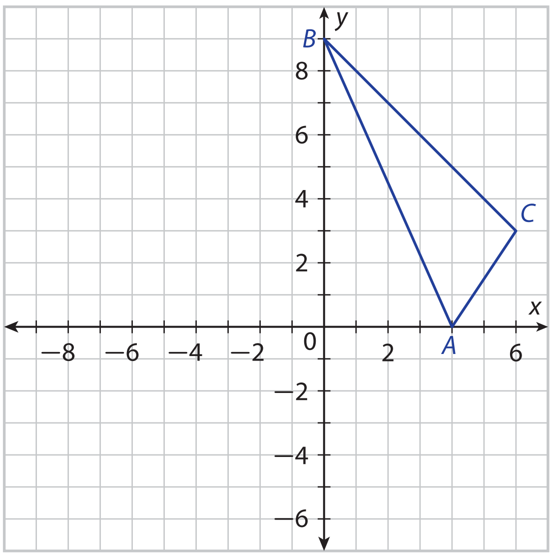


Copy $\triangle ABC$ on a coordinate grid. Then draw and label the final image of $\triangle ABC$ after the given sequence of transformations.

5. $(x, y) \to \left(x, \frac{1}{3}y\right) \to (-2x, -2y)$

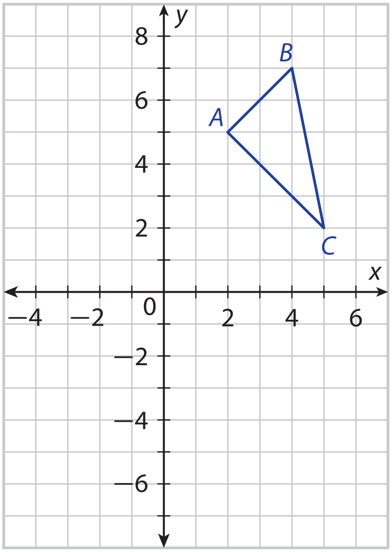


6. $(x, y) \to \left(-\frac{3}{2}x, \frac{2}{3}y\right) \to (x + 6, y - 4) \to \left(\frac{2}{3}x, -\frac{3}{2}y\right)$

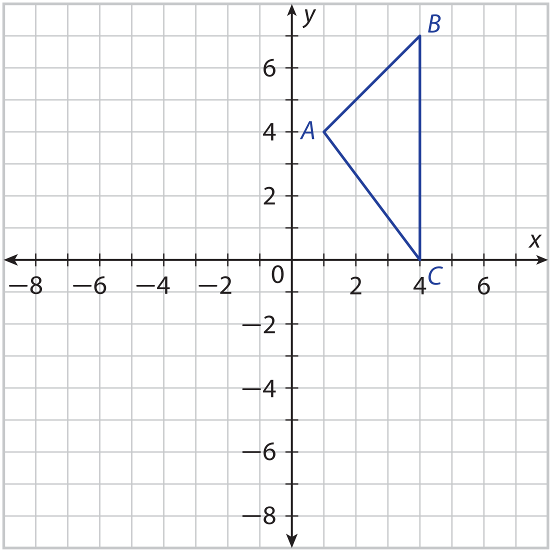


Predict the result of applying the sequence of transformations to the given figure.

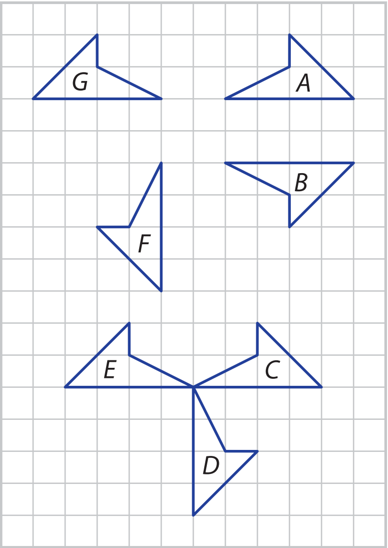
7. $\triangle ABC$ is translated along the vector $\langle -3, -1 \rangle$, reflected across the x-axis, and then reflected across the y-axis.



8. $\triangle ABC$ is translated along the vector $\langle -1, -3 \rangle$, rotated 180° about the origin, and then dilated by a factor of 2.



In Exercises 9–12, use the diagram. Complete each sentence with the letter of the correct image.



9. __?__ is the result of the sequence: *G* reflected over a vertical line and then a horizontal line.

10. __?__ is the result of the sequence: *D* rotated 90° clockwise around one of its vertices and then reflected over a horizontal line.

11. __?__ is the result of the sequence: *E* translated and then rotated 90° counterclockwise.

12. __?__ is the result of the sequence: *D* rotated 90° counterclockwise and then translated.

Write always, sometimes, or never to complete a true statement.

13. A combination of two rigid transformations on a preimage will __?__ produce the same image when taken in a different order.

14. A double rotation can __?__ be written as a single rotation.

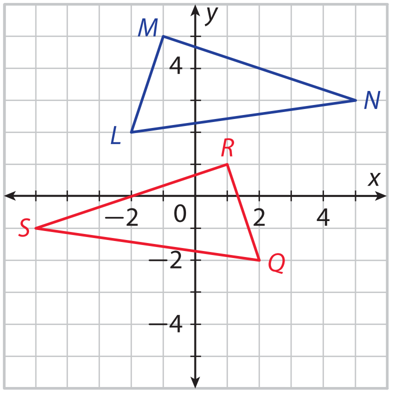
15. A sequence of a translation and a reflection __?__ has a point that does not change position.

16. A sequence of a reflection across the x-axis and then a reflection across the y-axis __?__ results in a 180° rotation of the preimage.

17. A sequence of rigid transformations will __?__ result in an image that is the same size and orientation as the preimage.

18. A sequence of a rotation and a dilation will __?__ result in an image that is the same size and orientation as the preimage.

19. $\triangle QRS$ is the image of $\triangle LMN$ under a sequence of transformations. Tell whether each of the following sequences can be used to create the image, $\triangle QRS$, from the preimage, $\triangle LMN$.



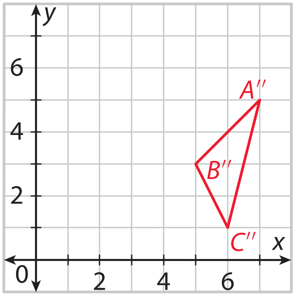
a. Reflect across the y-axis and then translate along the vector $\langle 0, -4 \rangle$.

b. Translate along the vector $\langle 0, -4 \rangle$ and then reflect across the y-axis.

c. Rotate 90° clockwise about the origin, reflect across the x-axis, and then rotate 90° counterclockwise about the origin.

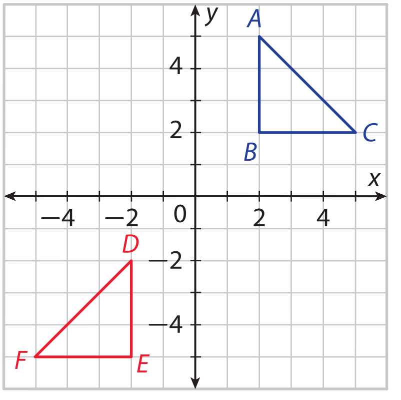
d. Rotate 180° about the origin, reflect across the x-axis, and then translate along the vector $\langle 0, -4 \rangle$.

20. A teacher gave students this puzzle: "I had a triangle with vertex A at (1, 4) and vertex B at (3, 2). After two rigid transformations, I had the image shown. Describe and show a sequence of transformations that will give this image from the preimage."



H.O.T. Focus on Higher Order Thinking

21. Analyze Relationships What two transformations would you apply to $\triangle ABC$ to get $\triangle DEF$? How could you express these transformations with a single mapping rule in the form of $(x, y) \rightarrow (?, ?)$?



22. **Multi-Step** Muralists will often make a scale drawing of an art piece before creating the large finished version. A muralist has sketched an art piece on a sheet of paper that is 3 feet by 4 feet.

a. If the final mural will be 39 feet by 52 feet, what is the scale factor for this dilation?

b. The owner of the wall has decided to only give permission to paint on the lower half of the wall. Can the muralist simply use the transformation $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$ in addition to the scale factor to alter the sketch for use in the allowed space? Explain.

23. **Communicate Mathematical Ideas** As a graded class activity, your teacher asks your class to reflect a triangle across the y-axis and then across the x-axis. Your classmate gets upset because he reversed the order of these reflections and thinks he will have to start over. What can you say to your classmate to help him?

Lesson Performance Task

The photograph shows an actual snowflake. Draw a detailed sketch of the "arm" of the snowflake located at the top left of the photo (10:00 on a clock face). Describe in as much detail as you can any translations, reflections, or rotations that you see.

Then describe how the entire snowflake is constructed, based on what you found in the design of one arm.